Essential norms of weighted composition operators between Hardy spaces H^p and H^q for $1 \le p, q \le \infty$

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Abstract

We complete the different cases remaining in the estimation of the essential norm of a weighted composition operator acting between the Hardy spaces H^p and H^q for $1 \le p, q \le \infty$. In particular we give some estimates for the cases $1 = p \le q \le \infty$ and $1 \le q < p \le \infty$.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} \mid \ |z| < 1\}$ denote the open unit disk in the complex plane. Given two analytic functions u and φ defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, one can define the *weighted composition operator* uC_φ that maps any analytic function f defined on $\mathbb{D}$ into the function $uC_\varphi(f) = u(f \circ \varphi)$. In [10], de Leeuw showed that the isometries in the Hardy space H^1 are weighted composition operators, while Forelli [8] obtained this result for the Hardy space H^p when $1 < p < \infty,\ p \neq 2$. Another example is the study of composition operators on the half-plane. A composition operator in a Hardy space of the half-plane is bounded if and only if a certain weighted composition operator is bounded on the Hardy space of the unit disk (see [13] and [14]).

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When $u \equiv 1$, we just have the composition operator C_φ. The continuity of these operators on the Hardy space H^p is ensured by the Littlewood's subordination principle, which says that $C_\varphi(f)$ belongs to H^p whenever $f \in H^p$ (see [4], Corollary 2.24). As a consequence, the condition $u \in H^\infty$ suffices for the boundedness of uC_φ on H^p. Considering the image of the constant functions, a necessary condition is that u belongs to H^p. Nevertheless a weighted composition operator needs not to be continuous on H^p, and it is easy to find examples where $uC_\varphi(H^p) \not\subseteq H^p$ (see Lemma 2.1 of [3] for instance).

In this note we deal with weighted composition operators between H^p and H^q for $1 \leq p, q \leq \infty$. Boundedness and compactness are characterized in [3] for $1 \leq p \leq q < \infty$ by means of Carleson measures, while essential norms of weighted composition operators are estimated in [5] for $1 < p \leq q < \infty$ by means of an integral operator. For the case $1 \leq q < p < \infty$, boundedness and compactness of uC_φ are studied in [5], and Gorkin and MacCluer in [9] gave an estimate of the essential norm of a composition operator acting between H^p and H^q.

The aim of this paper is to complete the different cases remaining in the estimation of the essential norm of a weighted composition operator. In section 2 and 3, we give an estimate of the essential norm of uC_φ acting between H^p and H^q when $p = 1$ and $1 \leq q < \infty$ and when $1 \leq p < \infty$ and $q = \infty$. Sections 4 and 5 are devoted to the case where $\infty \geq p > q \geq 1$.

Let $\overline{\mathbb{D}}$ be the closure of the unit disk $\mathbb{D}$ and $\mathbb{T} = \partial\mathbb{D}$ its boundary. We denote by $\mathrm{d}m = \mathrm{d}t/2\pi$ the normalised Haar measure on $\mathbb{T}$. If A is a Borel subset of $\mathbb{T}$, the notation $m(A)$ as well as $|A|$ will design the Haar measure of A. For $1 \leq p < \infty$, the Hardy space $H^p(\mathbb{D})$ is the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying the following condition

$$\|f\|_p = \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(r\zeta)|^p \,\mathrm{d}m(\zeta) \right)^{1/p} < \infty.$$

Endowed with this norm, $H^p(\mathbb{D})$ is a Banach space. The space $H^\infty(\mathbb{D})$ is consisting of every bounded analytic function on $\mathbb{D}$, and its norm is given by the supremum norm on $\mathbb{D}$.

We recall that any function $f \in H^p(\mathbb{D})$ can be extended on $\mathbb{T}$ to a function f^* by the following formula: $f^*(e^{i\theta}) = \lim_{r \nearrow 1} f(re^{i\theta})$. The limit exists almost everywhere by Fatou's theorem, and $f^* \in L^p(\mathbb{T})$. Moreover, $f \mapsto f^*$ is an into isometry from $H^p(\mathbb{D})$ to $L^p(\mathbb{T})$ whose image, denoted by $H^p(\mathbb{T})$ is the closure (weak-star closure for $p = \infty$) of the set of polynomials in $L^p(\mathbb{T})$. So

we can identify $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$, and we will use the notation H^p for both of these spaces. More on Hardy spaces can be found in [11] for instance.

The *essential norm* of an operator $T : X \to Y$, denoted $\|T\|_e$, is given by

$$\|T\|_e = \inf\{\|T - K\| \mid \ K \text{ is a compact operator from } X \text{ to } Y\}.$$

Observe that $\|T\|_e \leq \|T\|$, and $\|T\|_e$ is the norm of T seen as an element of the space $B(X,Y)/K(X,Y)$ where $B(X,Y)$ is the space of all bounded operators from X to Y and $K(X,Y)$ is the subspace consisting of all compact operators.

Notation: we will write $a \approx b$ whenever there exists two positive universal constants c and C such that $cb \leq a \leq Cb$. In the sequel, u will be a *non-zero analytic function* on $\mathbb{D}$ and φ will be a *non-constant analytic function* defined on $\mathbb{D}$ satisfying $\varphi(\mathbb{D}) \subset \mathbb{D}$.

2 $uC_\varphi \in B(H^1, H^q)$ for $1 \leq q < \infty$

Let us first start with a characterization of the boundedness of uC_φ acting between H^p and H^q:

Theorem 2.1 (see [5, Theorem 4]). *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Let $0 < p \leq q < \infty$. Then the weighted composition operator uC_φ is bounded from H^p to H^q if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{T}} |u(\zeta)|^q \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(\zeta)|^2}\right)^{q/p} \mathrm{d}m(\zeta) < \infty.$$

As a consequence uC_φ is a bounded operator as soon as uC_φ is uniformly bounded on the set $\{k_a^{1/p} \mid \ a \in \mathbb{D}\}$ where k_a is the normalized kernel defined by $k_a(z) = (1-|a|^2)/(1-\bar{a}z)^2,\ a \in \mathbb{D}$. Note that $k_a^{1/p} \in H^p$ and $\|k_a^{1/p}\|_p = 1$. These kernels play a crucial role in the estimation of the essential norm of a weighted composition operator:

Theorem 2.2 (see [5, Theorem 5]). *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Assume that the weighted composition operator uC_φ is bounded from H^p to H^q with $1 < p \leq q < \infty$. Then*

$$\|uC_\varphi\|_e \approx \limsup_{|a|\to 1^-} \left(\int_{\mathbb{T}} |u(\zeta)|^q \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(\zeta)|^2}\right)^{q/p} \mathrm{d}m(\zeta)\right)^{\frac{1}{q}}.$$

The aim of this section is to give the corresponding estimate for the case $p = 1$. We shall prove that the previous theorem is still valid for $p = 1$:

Theorem 2.3. *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that the weighted composition operator uC_φ is bounded from H^1 to H^q for a certain $1 \leq q < \infty$. Then we have*

$$\|uC_\varphi\|_e \approx \limsup_{|a|\to 1^-} \left(\int_{\mathbb{T}} |u(\zeta)|^q \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(\zeta)|^2} \right)^q \mathrm{d}m(\zeta) \right)^{\frac{1}{q}}.$$

Let us start with the upper estimate:

Proposition 2.4. *Let $uC_\varphi \in B(H^1, H^q)$ with $1 \leq q < \infty$. Then there exists a positive constant γ such that*

$$\|uC_\varphi\|_e \leq \gamma \limsup_{|a|\to 1^-} \left(\int_{\mathbb{T}} |u(\zeta)|^q \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(\zeta)|^2} \right)^q \mathrm{d}m(\zeta) \right)^{\frac{1}{q}}.$$

The main tool of the proof is the use of Carleson measures. Assume that μ is a finite positive Borel measure on $\overline{\mathbb{D}}$ and let $1 \leq p, q < \infty$. We say that μ is a (p, q)-Carleson measure if the embedding $J_\mu : f \in H^p \mapsto f \in L^q(\mu)$ is well defined. In this case, the closed graph theorem ensures that J_μ is continuous. In other words, μ is a (p, q)-Carleson measure if there exists a constant $\gamma_1 > 0$ such that for every $f \in H^p$,

$$\int_{\overline{\mathbb{D}}} |f(z)|^q \,\mathrm{d}\mu(z) \leq \gamma_1 \|f\|_p^q. \tag{2.1}$$

Let I be an arc in $\mathbb{T}$. By $S(I)$ we denote the Carleson window given by

$$S(I) = \{z \in \mathbb{D} \mid\ 1 - |I| \leq |z| < 1,\ z/|z| \in I\}.$$

Let us denote by $\mu_{\mathbb{D}}$ and $\mu_{\mathbb{T}}$ the restrictions of μ to $\mathbb{D}$ and $\mathbb{T}$ respectively. The following result is a version of a theorem of Duren (see [7], p.163) for measures on $\overline{\mathbb{D}}$:

Theorem 2.5 (see [1, Theorem 2.5])**.** *Let $1 \leq p < q < \infty$. A finite positive Borel measure μ on $\overline{\mathbb{D}}$ is a (p, q)-Carleson measure if and only if $\mu_{\mathbb{T}} = 0$ and there exists a constant $\gamma_2 > 0$ such that*

$$\mu_{\mathbb{D}}\left(S(I)\right) \leq \gamma_2 |I|^{q/p} \quad \textit{for any arc } I \subset \mathbb{T}. \tag{2.2}$$

Notice that the best constants γ_1 and γ_2 in (2.1) and (2.2) are comparable, meaning that there is a positive constant β independent of the measure μ such that $(1/\beta)\gamma_2 \le \gamma_1 \le \beta\gamma_2$.
The notion of Carleson measure was introduced by Carleson in [2] as a part of his work on the corona problem. He gave a characterization of measures μ on $\mathbb{D}$ such that H^p embeds continuously in $L^p(\mu)$.

Examples of such Carleson measures are provided by composition operators. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map and let $1 \le p, q < \infty$. The boundedness of the composition operator $C_\varphi : f \mapsto f \circ \varphi$ between H^p and H^q can be rephrased in terms of (p,q)-Carleson measures. Indeed, denote by m_φ the *pullback measure* of m by φ, which is the image of the Haar measure m of $\mathbb{T}$ under the map φ^*, defined by

$$m_\varphi(A) = m\left({\varphi^*}^{-1}(A)\right)$$

for every Borel subset A of $\overline{\mathbb{D}}$. Then

$$\|C_\varphi(f)\|_q^q = \int_{\mathbb{T}} |f \circ \varphi|^q \, \mathrm{d}m = \int_{\overline{\mathbb{D}}} |f|^q \, \mathrm{d}m_\varphi = \|J_{m_\varphi}(f)\|_q^q$$

for all $f \in H^p$. Thus C_φ maps H^p boundedly into H^q if and only if m_φ is a (p,q)-Carleson measure.
In the sequel we will denote by $r\mathbb{D}$ the open disk of radius r, in other words $r\mathbb{D} = \{z \in \mathbb{D} \mid \ |z| < r\}$ for $0 < r < 1$. We will need the following lemma concerning (p,q)-Carleson measures:

Lemma 2.6. *Take $0 < r < 1$ and let μ be a finite positive Borel measure on $\overline{\mathbb{D}}$. Let*

$$N_r^* := \sup_{|a| \ge r} \int_{\overline{\mathbb{D}}} |k_a(w)|^{\frac{q}{p}} \, \mathrm{d}\mu(w).$$

If μ is a (p,q)-Carleson measure for $1 \le p \le q < \infty$ then so is $\mu_r := \mu_{|_{\overline{\mathbb{D}} \setminus r\mathbb{D}}}$. Moreover one can find an absolute constant $M > 0$ satisfying $\|\mu_r\| \le M N_r^*$ where $\|\mu_r\| := \sup_{I \subset \mathbb{T}} \dfrac{\mu_r\big(S(I)\big)}{|I|^{q/p}}$.

We omit the proof of Lemma 2.6 here, which is a slight modification of the proof of Lemma 1 and Lemma 2 in [5] using Theorem 2.5.

In the proof of the upper estimate of Theorem 2.2 in [5], the authors use a decomposition of the identity on H^p of the form $I = K_N + R_N$ where K_N is the partial sum operator defined by $K_N\left(\sum_{n=0}^\infty a_n z^n\right) = \sum_{n=0}^N a_n z^n$, and they use the fact that (K_N) is a sequence of compact operators that is uniformly bounded in $B(H^p)$ and that R_N converges pointwise to zero on

H^p. Nevertheless the sequence (K_N) is not uniformly bounded in $B(H^1)$. In fact, (K_N) is uniformly bounded in $B(H^p)$ if and only if the Riesz projection $P : L^p \to H^p$ is bounded [15, Theorem 2], which occurs if and only if $1 < p < \infty$. Therefore we need to use a different decomposition for the case $p = 1$. Since K_N is the convolution operator by the Dirichlet kernel on H^p, we shall consider the Fejér kernel F_N of order N. Let us define $K_N : H^1 \to H^1$ to be the convolution operator associated to F_N that maps $f \in H^1$ to $K_N f = F_N * f \in H^1$ and $R_N = I - K_N$. Then $\|K_N\| \leq 1,\ K_N$ is compact and for every $f \in H^1,\ \|f - K_N f\|_1 \to 0$ following Fejér's theorem. If $f(z) = \sum_{n \geq 0} \hat{f}(n) z^n \in H^1$, then

$$K_N f(z) = \sum_{n=0}^{N-1} \left(1 - \frac{n}{N}\right) \hat{f}(n) z^n.$$

Lemma 2.7. *Let $1 \leq q < \infty$ and suppose that $uC_\varphi \in B(H^1, H^q)$. Then*

$$\|uC_\varphi\|_e \leq \liminf_N \|uC_\varphi R_N\|.$$

Proof.

$$\begin{aligned} \|uC_\varphi\|_e &= \|uC_\varphi K_N + uC_\varphi R_N\|_e \\ &= \|uC_\varphi R_N\|_e && \text{since } K_N \text{ is compact} \\ &\leq \|uC_\varphi R_N\| \end{aligned}$$

and the result follows taking the lower limit. □

We will need the following lemma for an estimation of the remainder R_N:

Lemma 2.8. *Let $\varepsilon > 0$ and $0 < r < 1$. Then $\exists N_0 = N_0(r) \in \mathbb{N},\ \forall N \geq N_0$,*

$$|R_N f(w)|^q < \varepsilon \|f\|_1^q,$$

for every $|w| < r$ and for every f in H^1.

Proof. Let $K_w(z) = 1/(1 - \bar{w} z),\ w \in \mathbb{D},\ z \in \mathbb{D}$. K_w is a bounded analytic function on $\mathbb{D}$. It is easy to see that for every $f \in H^1$,

$$\langle R_N f, K_w \rangle = \langle f, R_N K_w \rangle$$

where $|w| < r,\ N \geq 1$ and

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, \mathrm{d}\theta$$

for $f \in H^1$ and $g \in H^\infty$. Then we have $|R_N f(w)| = |\langle R_N f, K_w\rangle| = |\langle f, R_N K_w\rangle| \le \|f\|_1 \|R_N K_w\|_\infty$. Take $|w| < r$ and choose $N_0 \in \mathbb{N}$ so that for every $N \ge N_0$ one has $r^N \le \varepsilon^{1/q}(1-r)/2$ and $1/N \sum_{n=1}^{N-1} n r^n \le (1/2)\varepsilon^{1/q}$. Since

$$R_N K_w(z) = R_N\left(\sum_{n=0}^{\infty} \bar{w}^n z^n\right) = \sum_{n=0}^{N-1} \frac{n}{N} \bar{w}^n z^n + \sum_{n=N}^{\infty} \bar{w}^n z^n,$$

one has

$$\|R_N K_w\|_\infty < \frac{1}{N}\sum_{n=0}^{N-1} n r^n + \sum_{n=N}^{\infty} r^n \le \varepsilon^{1/q}.$$

Thus $|R_N f(w)|^q \le \varepsilon \|f\|_1^q$ for every f in H^1. □

Proof of Proposition 2.4. Denote by μ the measure which is absolutely continuous with respect to m and whose density is $|u|^q$, and let $\mu_\varphi = \mu \circ \varphi^{-1}$ be the pullback measure of μ by φ. Fix $0 < r < 1$. For every $f \in H^1$, we have

$$\begin{aligned} \|(uC_\varphi R_N)f\|_q^q &= \int_{\mathbb{T}} |u(\zeta)|^q \big|\big((R_N f)\circ\varphi\big)(\zeta)\big|^q \,\mathrm{d}m(\zeta) \\ &= \int_{\mathbb{T}} \big|\big((R_N f)\circ\varphi\big)(\zeta)\big|^q \,\mathrm{d}\mu(\zeta) \\ &= \int_{\overline{\mathbb{D}}} |R_N f(w)|^q \,\mathrm{d}\mu_\varphi(w) \\ &= \int_{\overline{\mathbb{D}}\setminus r\mathbb{D}} |R_N f(w)|^q \,\mathrm{d}\mu_\varphi(w) + \int_{r\mathbb{D}} |R_N f(w)|^q \,\mathrm{d}\mu_\varphi(w) \\ &= I_1(N,r,f) + I_2(N,r,f). \end{aligned} \tag{2.3}$$

Let us first show that $\lim_N \sup_{\|f\|_1=1} I_2(N,r,f) = 0$. For $\varepsilon > 0$, Lemma 2.8 gives us an integer $N_0(r)$ such that for every $N \ge N_0(r)$,

$$\begin{aligned} I_2(N,r,f) &= \int_{r\mathbb{D}} |R_N f(w)|^q \,\mathrm{d}\mu_\varphi(w) \\ &\le \varepsilon \|f\|_1^q \mu_\varphi(r\mathbb{D}) \\ &\le \varepsilon \|f\|_1^q \mu_\varphi(\overline{\mathbb{D}}) \\ &\le \varepsilon \|f\|_1^q \|u\|_q^q. \end{aligned}$$

So, r being fixed, we have $\lim_N \sup_{\|f\|_1=1} I_2(N,r,f) = 0$.

Now we need an estimate of $I_1(N,r,f)$. The continuity of $uC_\varphi : H^1 \to H^q$ ensures that μ_φ is a $(1,q)$-Carleson measure, and therefore $\mu_{\varphi,r} := \mu_{\varphi_{|\overline{\mathbb{D}}\setminus r\mathbb{D}}}$

is also a $(1,q)$-Carleson measure by using Lemma 2.6 for $p=1$. It follows that

$$\begin{aligned}\int_{\overline{\mathbb{D}}\backslash r\mathbb{D}} |R_N f(w)|^q \,\mathrm{d}\mu_{\varphi,r}(w) &\le \gamma_1 \|R_N f\|_1^q \\ &\le \beta \|\mu_{\varphi,r}\| \|R_N f\|_1^q \\ &\le 2^q \beta M N_r^* \|f\|_1^q\end{aligned}$$

using Lemma 2.6 and the fact that $\|R_N\| \le 1 + \|K_N\| \le 2$ for every $N \in \mathbb{N}$. We take the supremum over B_{H^1} and take the lower limit as N tends to infinity in (2.3) to obtain

$$\liminf_{N\to\infty} \|uC_\varphi R_N\|^q \le 2^q \beta M N_r^*.$$

Now as r goes to 1 we have:

$$\begin{aligned}\lim_{r\to 1} N_r^* &= \limsup_{|a|\to 1^-} \int_{\overline{\mathbb{D}}} |k_a(w)|^q \,\mathrm{d}\mu_\varphi(w) \\ &= \limsup_{|a|\to 1^-} \int_{\mathbb{T}} |u(\zeta)|^q \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(\zeta)|^2}\right)^q \,\mathrm{d}m(\zeta)\end{aligned}$$

and we obtain the estimate announced using Lemma 2.7. $\square$

Now let us turn to the lower estimate in Theorem 2.2. Let $1 \le q < \infty$. Consider F_N the Fejér kernel of order N, and define $K_N : H^q \to H^q$ the convolution operator associated to F_N and $R_N = I - K_N$. Then $(K_N)_N$ is a sequence of uniformly bounded compact operators in $B(H^q)$, and $\|R_N f\|_q \to 0$ for all $f \in H^q$.

Lemma 2.9. *There exists $0 < \gamma \le 2$ such that whenever uC_φ is a bounded operator from H^1 to H^q with $1 \le q < \infty$, one has*

$$\frac{1}{\gamma} \limsup_N \|R_N uC_\varphi\| \le \|uC_\varphi\|_e.$$

Proof. Take $K \in B(H^1, H^q)$ a compact operator. Since (K_N) is uniformly bounded, one can find $\gamma > 0$ satisfying $\|R_N\| \le 1 + \|K_N\| \le \gamma$ for all $N > 0$, and we have:

$$\begin{aligned}\|uC_\varphi + K\| &\ge \frac{1}{\gamma} \|R_N(uC_\varphi + K)\| \\ &\ge \frac{1}{\gamma}\|R_N uC_\varphi\| - \frac{1}{\gamma}\|R_N K\|.\end{aligned}$$

Now use the fact that (R_N) goes pointwise to zero in H^q, and consequently (R_N) converges strongly to zero over the compact set $\overline{K(B_{H^1})}$ as N goes to infinity. It follows that $\|R_N K\| \underset{N}{\longrightarrow} 0$, and

$$\|uC_\varphi + K\| \geq \frac{1}{\gamma} \limsup_N \|R_N uC_\varphi\|$$

for every compact operator $K : H^1 \to H^q$. □

Proposition 2.10. *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Assume that $uC_\varphi \in B(H^1, H^q)$ with $1 \leq q < \infty$. Then*

$$\|uC_\varphi\|_e \geq \frac{1}{\gamma} \limsup_{|a|\to 1^-} \left(\int_{\mathbb{T}} |u(\zeta)|^q \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(\zeta)|^2} \right)^q \,\mathrm{d}m(\zeta) \right)^{\frac{1}{q}}.$$

Proof. Since k_a is a unit vector in H^1,

$$\|R_N uC_\varphi\| = \|uC_\varphi - K_N uC_\varphi\| \geq \|uC_\varphi k_a\|_q - \|K_N uC_\varphi k_a\|_q. \tag{2.4}$$

First case: $q > 1$

Since (k_a) converges to zero for the topology of uniform convergence on compact sets in $\mathbb{D}$ as $|a|$ goes to 1, so does $uC_\varphi(k_a)$. The topology of uniform convergence on compact sets in $\mathbb{D}$ and the weak topology agree on H^q, therefore it follows that $uC_\varphi(k_a)$ goes to zero for the weak topology in H^q as $|a|$ goes to 1. Since K_N is a compact operator, it is completely continuous and carries weak-null sequences to norm-null sequences. So $\|K_N\big(uC_\varphi(k_a)\big)\|_q \to 0$ when $|a| \to 1$, and

$$\|R_N uC_\varphi\| \geq \limsup_{|a|\to 1^-} \|uC_\varphi(k_a)\|_q.$$

Taking the upper limit as $N \to \infty$, we obtain the result using Lemma 2.9. For the second case we will need the following computational lemma:

Lemma 2.11. *Let φ be an analytic self-map of $\mathbb{D}$. Take $a \in \mathbb{D}$ and $N \geq 1$ an integer. Denote by $\alpha_p(a)$ the p-th Fourier coefficient of $C_\varphi\left(k_a/(1-|a|^2)\right)$, so that for every $z \in \mathbb{D}$ we have*

$$k_a\big(\varphi(z)\big) = (1-|a|^2) \sum_{p=0}^{\infty} \alpha_p(a) z^p.$$

Then there exists a positive constant $M = M(N) > 0$ depending on N such that $|\alpha_p(a)| \leq M$ for every $p \leq N$ and every $a \in \mathbb{D}$.

Proof. Write $\varphi(z) = a_0 + \psi(z)$ with $a_0 = \varphi(0) \in \mathbb{D}$ and $\psi(0) = 0$. If we develop $k_a(z)$ as a Taylor series and replace z by $\varphi(z)$ we obtain:

$$k_a\big(\varphi(z)\big) = (1 - |a|^2) \sum_{n=0}^{\infty} (n+1)(\bar{a})^n \varphi(z)^n.$$

Then

$$\begin{aligned}
\alpha_p(a) &= \left\langle \sum_{n=0}^{\infty} (n+1)(\bar{a})^n \varphi(z)^n, z^p \right\rangle \\
&= \sum_{n=0}^{\infty} (n+1)(\bar{a})^n \sum_{j=0}^{n} \binom{n}{j} a_0^{n-j} \left\langle \psi(z)^j, z^p \right\rangle .
\end{aligned}$$

where $\langle f, g\rangle = \int_{\mathbb{T}} f\bar{g}\ \mathrm{d}m$. Note that $\langle \psi(z)^j, z^p\rangle = 0$ if $j > p$ since $\psi(0) = 0$, and consequently

$$\begin{aligned}
\alpha_p(a) &= \sum_{n=0}^{\infty} (n+1)(\bar{a})^n \sum_{j=0}^{\min(n,p)} \binom{n}{j} a_0^{n-j} \left\langle \psi(z)^j, z^p \right\rangle \\
&= \sum_{j=0}^{p} \sum_{n=j}^{\infty} (n+1)(\bar{a})^n \binom{n}{j} a_0^{n-j} \left\langle \psi(z)^j, z^p \right\rangle \\
&= \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle \sum_{n=j}^{\infty} (n+1)(\bar{a})^n \binom{n}{j} a_0^{n-j}.
\end{aligned}$$

In the case where $a_0 \neq 0$ we obtain

$$\begin{aligned}
\alpha_p(a) &= \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle a_0^{-j} \sum_{n=j}^{\infty} (n+1) \binom{n}{j} (\bar{a}a_0)^n \\
&= \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle a_0^{-j} \frac{(j+1)(\bar{a}a_0)^j}{(1-\bar{a}a_0)^{j+2}} \\
&= \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle \frac{(j+1)(\bar{a})^j}{(1-\bar{a}a_0)^{j+2}}
\end{aligned}$$

using the following equalities for $x = \bar{a}a_0 \in \mathbb{D}$:

$$\sum_{n=j}^{\infty} (n+1) \binom{n}{j} x^n = \left(\sum_{n=j}^{\infty} \binom{n}{j} x^{n+1} \right)' = \left(\frac{x^{j+1}}{(1-x)^{j+1}} \right)' = \frac{(j+1)x^j}{(1-x)^{j+2}}$$

Note that the last expression obtained for $\alpha_p(a)$ is also valid for $a_0 = 0$.

Thus, for $0 \le p \le N$ we have the following estimates:

$$\begin{aligned}
|\alpha_p(a)| &\le \sum_{j=0}^{p} |\left\langle \psi(z)^j, z^p \right\rangle| \frac{j+1}{(1-|a_0|)^{j+2}} \\
&\le \sum_{j=0}^{p} \|\psi^j\|_\infty \frac{N+1}{(1-|a_0|)^{N+2}} \\
&\le \frac{(N+1)^2}{(1-|a_0|)^{N+2}} \max_{0\le j\le N} \|\psi^j\|_\infty \\
&\le M,
\end{aligned}$$

where M is a constant independent from a. ◻

Second case: $q = 1$

In this case, it is no longer for the weak topology but for the weak-star topology of H^1 that $uC_\varphi(k_a)$ tends to zero when $|a| \to 1$. Nevertheless, it is still true that $\|K_N uC_\varphi(k_a)\|_1 \to 0$ as $|a| \to 1$. Indeed if $f(z) = \sum_{n\ge 0} \hat{f}(n) z^n \in H^1$, then

$$K_N f(z) = \sum_{n=0}^{N-1} \left(1 - \frac{n}{N}\right) \hat{f}(n) z^n.$$

We have the following development:

$$k_a\big(\varphi(z)\big) = (1-|a|^2) \sum_{n=0}^{\infty} \alpha_n(a) z^n.$$

Denote by u_n the n-th Fourier coefficient of u, so that

$$uC_\varphi(k_a)(z) = (1-|a|^2) \sum_{n=0}^{\infty} \left(\sum_{p=0}^{n} \alpha_p(a) u_{n-p} \right) z^n, \ \forall z \in \mathbb{D}.$$

It follows that

$$\|K_N uC_\varphi(k_a)\|_1 \le (1-|a|^2) \sum_{n=0}^{N-1} \left(1 - \frac{n}{N}\right) \left| \sum_{p=0}^{n} \alpha_p(a) u_{n-p} \right| \|z^n\|_1.$$

Now using estimates from Lemma 2.11, one can find a constant $M > 0$ independent from a such that $|\alpha_p(a)| \le M$ for every $a \in \mathbb{D}$ and $0 \le p \le N-1$. Use the fact that $\|z^n\|_1 = 1$ and $|u_p| \le \|u\|_1$ to deduce that there is a constant $M' > 0$ independent from a such that

$$\|K_N uC_\varphi(k_a)\|_1 \le M'(1-|a|^2)\|u\|_1$$

for all $a \in \mathbb{D}$. Thus $K_N uC_\varphi(k_a)$ converges to zero in H^1 when $|a| \to 1$, and take the upper limit in 2.4 when a tends to 1^- to obtain

$$\|R_N uC_\varphi\| \geq \limsup_{|a|\to 1} \|uC_\varphi(k_a)\|_1, \quad \forall N \geq 0.$$

We conclude with Lemma 2.9 and observe that $\gamma = \sup \|R_N\| \leq 2$ since $\|R_N\| \leq 1 + \|K_N\| \leq 2$. □

3 $uC_\varphi \in B(H^p, H^\infty)$ for $1 \leq p < \infty$

Let u be a bounded analytic function. Characterizations of boundedness and compactness of uC_φ as a linear map between H^p and H^∞ have been studied in [3] for $p \geq 1$. Indeed,

$$uC_\varphi \in B(H^p, H^\infty) \text{ if and only if } \sup_{z\in\mathbb{D}} \frac{|u(z)|^p}{1-|\varphi(z)|^2} < \infty$$

and

$$uC_\varphi \text{ is compact if and only if } \|\varphi\|_\infty < 1 \text{ or } \lim_{|\varphi(z)|\to 1} \frac{|u(z)|^p}{1-|\varphi(z)|^2} = 0.$$

In the case where $\|\varphi\|_\infty = 1$ we let

$$M_\varphi(u) = \limsup_{|\varphi(z)|\to 1} \frac{|u(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}}.$$

As regarding Theorem 1.7 in [12], it seems reasonable to think that the essential norm of uC_φ is equivalent to the quantity $M_\varphi(u)$. We first have a majorization:

Proposition 3.1. *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that uC_φ is a bounded operator from H^p to H^∞, where $1 \leq p < \infty$ and that $\|\varphi\|_\infty = 1$. Then*

$$\|uC_\varphi\|_e \leq 2M_\varphi(u).$$

Proof. Let ε be a real positive number, and pick $r < 1$ satisfying

$$\sup_{|\varphi(z)|\geq r} \frac{|u(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} \leq M_\varphi(u) + \varepsilon.$$

We approximate uC_φ by $uC_\varphi K_N$ where $K_N : H^p \to H^p$ is the convolution operator by the Fejér kernel of order N, where N is chosen so that $|R_N f(w)| < \varepsilon \|f\|_1$ for every $f \in H^1$ and every $|w| < r$ (Lemma 2.8 for

$q=1$). We want to show that $\|uC_\varphi - uC_\varphi K_N\| = \|uC_\varphi R_N\| \le \max(2M_\varphi(u)+2\varepsilon, \varepsilon\|u\|_\infty)$, which will prove our assertion. If f is a unit vector in H^p, then the norm of $uC_\varphi R_N(f)$ is equal to

$$\max\left(\sup_{|\varphi(z)|\ge r}|u(z)(R_N f)\circ\varphi(z)|, \sup_{|\varphi(z)|<r}|u(z)(R_N f)\circ\varphi(z)|\right).$$

We want to estimate the first term. If $\omega\in\mathbb{D}$, we denote by δ_ω the linear functional on H^p defined by $\delta_\omega(f)=f(\omega)$. Then $\delta_\omega\in(H^p)^*$ and $\|\delta_w\|_{(H^p)^*} = 1/(1-|w|^2)^{1/p}$ for every $w\in\mathbb{D}$. Therefore

$$\begin{aligned}\sup_{|\varphi(z)|\ge r}|u(z)(R_N f)\circ\varphi(z)| &\le \sup_{|\varphi(z)|\ge r}|u(z)|\|\delta_{\varphi(z)}\|_{(H^p)^*}\|R_N f\|_p\\ &\le 2\sup_{|\varphi(z)|\ge r}\frac{|u(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}}\\ &\le 2\left(M_\varphi(u)+\varepsilon\right),\end{aligned}$$

using the fact that $\|R_N f\|_p\le 2$.
For the second term, since $|\varphi(z)|<r$ we have

$$|u(z)R_N f\left(\varphi(z)\right)| \le \|u\|_\infty|R_N f\left(\varphi(z)\right)| \le \varepsilon\|u\|_\infty\|f\|_1 \le \varepsilon\|u\|_\infty$$

which ends the proof. □

On the other hand, we have the lower estimate:

Proposition 3.2. *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$ satisfying $\|\varphi\|_\infty=1$. Suppose that uC_φ is a bounded operator from H^p to H^∞, where $1\le p<\infty$. Then*

$$\frac{1}{2}M_\varphi(u)\le\|uC_\varphi\|_e.$$

Proof. Assume that uC_φ is not compact, implying $M_\varphi(u)>0$. Let (z_n) be a sequence in $\mathbb{D}$ satisfying

$$\lim_n|\varphi(z_n)|=1 \quad\text{and}\quad \lim_n\frac{|u(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{1}{p}}}=M_\varphi(u).$$

Consider the sequence (f_n) defined by

$$f_n(z)=k_{\varphi(z_n)}(z)^{1/p}=\frac{(1-|\varphi(z_n)|^2)^{\frac{1}{p}}}{\left(1-\overline{\varphi(z_n)}z\right)^{\frac{2}{p}}}.$$

Each f_n is a unit vector of H^p. Let $K: H^p\to H^\infty$ be a compact operator.

First case: $p > 1$

Since the sequence (f_n) converges to zero for the weak topology of H^p and K is completely continuous, the sequence (Kf_n) converges to zero for the norm topology in H^∞. Use that $\|uC_\varphi + K\| \geq \|uC_\varphi(f_n)\|_\infty - \|Kf_n\|_\infty$ and take the upper limit when n tends to infinity to obtain

$$\begin{aligned} \|uC_\varphi + K\| &\geq \limsup_n \|uC_\varphi(f_n)\|_\infty \\ &\geq \limsup_n |u(z_n)|\, |f_n(\varphi(z_n))| \\ &\geq \limsup_n \frac{|u(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{1}{p}}} \\ &\geq M_\varphi(u). \end{aligned}$$

Second case: $p = 1$

Let $\varepsilon > 0$. Since the sequence (f_n) is no longer weakly convergent to zero in H^1, we cannot assert that $(Kf_n)_n$ goes to zero in H^∞. Nevertheless, passing to subsequences, one can assume that $(Kf_{n_k})_k$ converges in H^∞, and hence is a Cauchy sequence. So we can find an integer $N > 0$ such that for every k and m greater than N we have $\|Kf_{n_k} - Kf_{n_m}\| < \varepsilon$. We deduce that

$$\begin{aligned} \|uC_\varphi + K\| &\geq \left\| (uC_\varphi + K)\left(\frac{f_{n_k} - f_{n_m}}{2}\right)\right\|_\infty \\ &\geq \frac{1}{2}\|uC_\varphi(f_{n_k} - f_{n_m})\|_\infty - \frac{\varepsilon}{2} \\ &\geq \frac{1}{2}|u(z_{n_k})|\, |f_{n_k}(\varphi(z_{n_k})) - f_{n_m}(\varphi(z_{n_k}))| - \frac{\varepsilon}{2} \\ &\geq \frac{|u(z_{n_k})|}{2\,(1-|\varphi(z_{n_k})|^2)} - \frac{|u(z_{n_k})|\,(1-|\varphi(z_{n_m})|^2)}{2\left|1-\overline{\varphi(z_{n_m})}\varphi(z_{n_k})\right|^2} - \frac{\varepsilon}{2} \end{aligned}$$

Now take the upper limit as m goes to infinity (k being fixed) and recall that $\lim_m |\varphi(z_{n_m})| = 1$ and $|\varphi(z_{n_k})| < 1$ to obtain

$$\|uC_\varphi + K\| \geq \frac{|u(z_{n_k})|}{2\,(1-|\varphi(z_{n_k})|^2)} - \frac{\varepsilon}{2}$$

for every $k \geq N$. It remains to make k tend to infinity to have

$$\|uC_\varphi + K\| \geq \frac{1}{2}M_\varphi(u) - \frac{\varepsilon}{2}.$$

$\square$

Combining Proposition 3.1 and Proposition 3.2 we obtain the following estimate:

Theorem 3.3. *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$ satisfying $\|\varphi\|_\infty = 1$. Suppose that uC_φ is a bounded operator from H^p to H^∞, where $1 \leq p < \infty$. Then $\|uC_\varphi\|_e \approx M_\varphi(u)$. More precisely, we have the following inequalities:*

$$\frac{1}{2}M_\varphi(u) \leq \|uC_\varphi\|_e \leq 2M_\varphi(u).$$

Note that if $p > 1$ one can replace the constant $1/2$ by 1.

4 $uC_\varphi \in B(H^\infty, H^q)$ for $\infty > q \geq 1$

In this setting, boundedness of the weighted composition operator uC_φ is equivalent to saying that u belongs to H^q, and uC_φ is compact if and only if $u = 0$ or $|E_\varphi| = 0$ where $E_\varphi = \{\zeta \in \mathbb{T} \mid \varphi^*(\zeta) \in \mathbb{T}\}$ is the extremal set of φ (see [3]). We give here some estimates of the essential norm of uC_φ that appear in [9] for the special case of composition operators:

Theorem 4.1. *Let $u \in H^q$, with $\infty > q \geq 1$ and φ be an analytic self-map of $\mathbb{D}$. Then $\|uC_\varphi\|_e \approx \left(\int_{E_\varphi} |u(\zeta)|^q \,\mathrm{d}m(\zeta)\right)^{\frac{1}{q}}$. More precisely,*

$$\frac{1}{2}\left(\int_{E_\varphi} |u(\zeta)|^q \,\mathrm{d}m(\zeta)\right)^{\frac{1}{q}} \leq \|uC_\varphi\|_e \leq 2\left(\int_{E_\varphi} |u(\zeta)|^q \,\mathrm{d}m(\zeta)\right)^{\frac{1}{q}}.$$

We start with the upper estimate:

Proposition 4.2. *Let $u \in H^q$, with $\infty > q \geq 1$ and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|uC_\varphi\|_e \leq 2\bigg(\int_{E_\varphi} |u(\zeta)|^q \,\mathrm{d}m(\zeta)\bigg)^{\frac{1}{q}}.$$

Proof. Take $0 < r < 1$. Since $\|r\varphi\|_\infty \leq r < 1$, the set $E_{r\varphi}$ is empty and therefore the operator $uC_{r\varphi}$ is compact. Thus $\|uC_\varphi\|_e \leq \|uC_\varphi - uC_{r\varphi}\|$. But

$$\|uC_\varphi - uC_{r\varphi}\|^q = \sup_{\|f\|_\infty \leq 1} \int_{\mathbb{T}} |u(\zeta)|^q \big|f\big(\varphi(\zeta)\big) - f\big(r\varphi(\zeta)\big)\big|^q \,\mathrm{d}m(\zeta). \tag{4.1}$$

If $|E_\varphi| = 1$ then the integral in (4.1) coincides with

$$\int_{E_\varphi} |u(\zeta)|^q \big|f\big(\varphi(\zeta)\big) - f\big(r\varphi(\zeta)\big)\big|^q \,\mathrm{d}m(\zeta)$$

which is less than $2^q \int_{E_\varphi} |u(\zeta)|^q \,\mathrm{d}m(\zeta)$. If $|E_\varphi| < 1$ we let $F_\varepsilon = \{\zeta \in \mathbb{T} \mid |\varphi^*(\zeta)| < 1 - \varepsilon\}$ for $\varepsilon > 0$, which is a nonempty set for ε sufficiently small.

(Let us mention here that an element $\zeta \in \mathbb{T}$ needs not to satisfy neither $\zeta \in E_\varphi$ nor $\zeta \in \bigcup_{\varepsilon>0} F_\varepsilon$. It can happen that the radial limit $\varphi^*(\zeta)$ does not exist, but this happens only for ζ belonging to a set of measure zero). We will use the pseudohyperbolic distance ρ defined for z and w in the unit disk by $\rho(z,w) = |z-w|/|1-\bar{w}z|$. The Pick-Schwarz's theorem ensures that $\rho\big(f(z), f(w)\big) \leq \rho(z,w)$ for every function $f \in B_{H^\infty}$. As a consequence the inequality $|f(z)-f(w)| \leq 2\rho(z,w)$ holds for every w and z in $\mathbb{D}$.
If ζ is an element of F_ε then

$$\rho\big(\varphi(\zeta), r\varphi(\zeta)\big) = \frac{(1-r)|\varphi(\zeta)|}{1-r|\varphi(\zeta)|^2} \leq \frac{1-r}{1-r(1-\varepsilon)^2}.$$

One can choose $0<r<1$ satisfying $\sup_{F_\varepsilon} \rho\big(\varphi(\zeta), r\varphi(\zeta)\big) < \varepsilon/2$, and therefore

$$\big|f\big(\varphi(\zeta)\big) - f\big(r\varphi(\zeta)\big)\big| \leq 2\sup_{F_\varepsilon} \rho\big(\varphi(\zeta), r\varphi(\zeta)\big) \leq \varepsilon$$

for all $\zeta \in F_\varepsilon$ and for every function f in the closed unit ball of H^∞. It follows from these estimates and (4.1) that

$$\begin{aligned}\|uC_\varphi - uC_{r\varphi}\|^q &\leq \sup_{\|f\|_\infty \leq 1} \left(\int_{F_\varepsilon} |u(\zeta)|^q \varepsilon^q \,\mathrm{d}m(\zeta) + \int_{\mathbb{T}\setminus F_\varepsilon} 2^q |u(\zeta)|^q \,\mathrm{d}m(\zeta) \right) \\ &\leq \varepsilon^q \|u\|_q^q + 2^q \int_{\mathbb{T}\setminus F_\varepsilon} |u(\zeta)|^q \,\mathrm{d}m(\zeta).\end{aligned}$$

Make ε tend to zero to deduce the upper estimate. □

Let us turn to the lower estimate:

Proposition 4.3. *Suppose that φ is an analytic self-map of $\mathbb{D}$ and $u \in H^q$ with $\infty > q \geq 1$. Then*

$$\|uC_\varphi\|_e \geq \frac{1}{2} \left(\int_{E_\varphi} |u(\zeta)|^q \,\mathrm{d}m(\zeta) \right)^{\frac{1}{q}}.$$

Proof. Take a compact operator $K \in B(H^\infty, H^q)$. Since the sequence $(z^n)_{n\in\mathbb{N}}$ is bounded in H^∞, there exists an increasing sequence of integers $(n_k)_{k\geq 0}$ such that $\big(K(z^{n_k})\big)_{k\geq 0}$ converges in H^q. For any $\varepsilon > 0$ one can find $N \in \mathbb{N}$ such that for every $k, m \geq N$ we have $\|Kz^{n_k} - Kz^{n_m}\|_q < \varepsilon$. If $0<r<1$, we let $g_r(z) = g(rz)$ for a function g defined on $\mathbb{D}$. Take $k \geq N$. Then there exists $0<r<1$ such that

$$\|\left(u\varphi^{n_k}\right)_r\|_q \geq \|u\varphi^{n_k}\|_q - \varepsilon.$$

For all $m \geq N$ we have

$$\begin{aligned}
\|uC_\varphi + K\| &\geq \left\| (uC_\varphi + K)\left(\frac{z^{n_k} - z^{n_m}}{2}\right)\right\|_q \\
&\geq \frac{1}{2}\left\|u\left(\varphi^{n_k} - \varphi^{n_m}\right)\right\|_q - \frac{\varepsilon}{2} \\
&\geq \frac{1}{2}\left\|(u\varphi^{n_k})_r - (u\varphi^{n_m})_r\right\|_q - \frac{\varepsilon}{2} \\
&\geq \frac{1}{2}\left(\left\|(u\varphi^{n_k})_r\right\|_q - \left\|(u\varphi^{n_m})_r\right\|_q\right) - \frac{\varepsilon}{2} \\
&\geq \frac{1}{2}\left(\left\|u\varphi^{n_k}\right\|_q - \left\|(u\varphi^{n_m})_r\right\|_q\right) - \varepsilon.
\end{aligned}$$

Let us make m tend to infinity, keeping in mind that $0 < r < 1$ and $\|\varphi_r\|_\infty < 1$:

$$\left\|\left(u\varphi^{n_m}\right)_r\right\|_q \leq \|u\|_q \|(\varphi_r)^{n_m}\|_\infty \leq \|u\|_q \|\varphi_r\|_\infty^{n_m} \xrightarrow[m]{} 0.$$

Thus $\|uC_\varphi + K\| \geq (1/2)\|u\varphi^{n_k}\|_q - \varepsilon$ for all $k \geq N$. We conclude noticing that

$$\|u\varphi^{n_k}\|_q = \left(\int_{\mathbb{T}} |u(\zeta)\varphi(\zeta)^{n_k}|^q \,\mathrm{d}m(\zeta)\right)^{\frac{1}{q}} \xrightarrow[k]{} \left(\int_{E_\varphi} |u(\zeta)|^q \,\mathrm{d}m(\zeta)\right)^{\frac{1}{q}}.$$

□

5 $uC_\varphi \in B(H^p, H^q)$ for $\infty > p > q \geq 1$

In [9], the authors give an estimate of the essential norm of a composition operator between H^p and H^q for $1 < q < p < \infty$. The proof makes use of the Riesz projection from L^q onto H^q, which is a bounded operator for $1 < q < \infty$. Since it is not bounded from L^1 to H^1 (H^1 is not even complemented in L^1) there is no way to use a similar argument. So we need a different approach to get some estimates for $q = 1$. A solution is to make use of Carleson measures. First, we give a characterization of the boundedness of uC_φ in terms of a Carleson measure. In the case where $p > q$, Carleson measures on $\overline{\mathbb{D}}$ are characterized in [1]. Denote by $\Gamma(\zeta)$ the Stolz domain generated by $\zeta \in \mathbb{T}$, *i.e.* the interior of the convex hull of the set $\{\zeta\} \cup (\alpha\mathbb{D})$, where $0 < \alpha < 1$ is arbitrary but fixed.

Theorem 5.1 (see [1, Theorem 2.2]). *Let μ be a measure on $\overline{\mathbb{D}}$, $1 \leq q < p < \infty$ and $s = p/(p-q)$. Then μ is a (p,q)-Carleson measure on $\overline{\mathbb{D}}$*

if and only if $\zeta \mapsto \int_{\Gamma(\zeta)} \dfrac{\mathrm{d}\mu(z)}{1-|z|^2}$ *belongs to* $L^s(\mathbb{T})$ *and* $\mu_{\mathbb{T}} = F\mathrm{d}m$ *for a function* $F \in L^s(\mathbb{T})$.

This leads to a characterization of the continuity of a weighted composition operator between H^p and H^q:

Corollary 5.2. *Let* u *be an analytic function on* $\mathbb{D}$ *and* φ *an analytic self-map of* $\mathbb{D}$. *For* $1 \le q < p < \infty$, *the weighted composition operator* $uC_\varphi : H^p \to H^q$ *is bounded if and only if* $G : \zeta \in \mathbb{T} \mapsto G(\zeta) = \int_{\Gamma(\zeta)} \frac{\mathrm{d}\mu_\varphi(z)}{1-|z|^2}$ *belongs to* $L^s(\mathbb{T})$ *for* $s = p/(p-q)$ *and* $\mu_{\varphi_{|\mathbb{T}}} = F\mathrm{d}m$ *for a certain* $F \in L^s(\mathbb{T})$, *where* $\mathrm{d}\mu = |u|^q \mathrm{d}m$ *and* $\mu_\varphi = \mu \circ \varphi^{-1}$ *is the pullback measure of* μ *by* φ.

Proof. uC_φ is a bounded operator if and only if there exists $\gamma > 0$ such that for any $f \in H^p$, $\int_{\mathbb{T}} |u(\zeta)|^q \, |f \circ \varphi(\zeta)|^q \ \mathrm{d}m(\zeta) \le \gamma \|f\|_p^q$, which is equivalent (via a change of variables) to $\int_{\overline{\mathbb{D}}} |f(z)|^q \ \mathrm{d}\mu_\varphi(z) \le \gamma \|f\|_p^q$ for every $f \in H^p$. This exactly means that μ_φ is a (p,q)-Carleson measure. This is equivalent by Theorem 5.1 to the condition announced. □

If $f \in H^p$, the Hardy-Littlewood maximal nontangential function Mf is defined by $Mf(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|$ for $\zeta \in \mathbb{T}$. For $1 < p < \infty$, M is a bounded operator from H^p to L^p and we will denote its norm by $\|M\|_p$. The following lemma is the analogue version of Lemma 2.6 for the case $p > q$.

Lemma 5.3. *Let* μ *be a positive Borel measure on* $\overline{\mathbb{D}}$. *Assume that* μ *is a* (p,q)*-Carleson measure for* $1 \le q < p < \infty$. *Let* $0 < r < 1$ *and* $\mu_r := \mu_{|\overline{\mathbb{D}} \setminus r\mathbb{D}}$. *Then* μ_r *is a* (p,q)*-Carleson measure, and there exists a positive constant* γ *such that for every* $f \in H^p$,

$$\int_{\overline{\mathbb{D}}} |f(z)|^q \ \mathrm{d}\mu_r(z) \le (\|F\|_s + \gamma \|M\|_p^q \|\widetilde{G_r}\|_s) \|f\|_p^q$$

where $\mathrm{d}\mu_{\mathbb{T}} = F\mathrm{d}m$ *and* $\widetilde{G_r}(\zeta) = \int_{\Gamma(\zeta)} \frac{\mathrm{d}\mu_r(z)}{1-|z|^2}$. *In addition,* $\|\widetilde{G_r}\|_s \to 0$ *as* $r \to 1$.

We use the notation $\widetilde{G_r}$ to avoid any confusion with the notation introduced before for φ and its radial function φ_r.

Proof. Being a (p,q)-Carleson measure only depends on the ratio p/q (see [1, Lemma 2.1]), so we have to show that μ_r is a $(p/q,1)$-Carleson measure. From the definition it is clear that $\widetilde{G_r} \le G \in L^s(\mathbb{T})$. Moreover $\mathrm{d}\mu_{r_{|\mathbb{T}}} =$

$d\mu_{\mathbb{T}} = F dm \in L^s(\mathbb{T})$. Corollary 5.2 ensures the fact that μ_r is a (p,q)-Carleson measure.
Let f be in H^p. Then

$$\begin{aligned}\int_{\mathbb{T}} |f(\zeta)|^q \, d\mu_r(\zeta) &= \int_{\mathbb{T}} |f(\zeta)|^q \, d\mu(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^q F(\zeta)\, dm(\zeta) \\ &\le \left(\int_{\mathbb{T}} |f(\zeta)|^p \, dm(\zeta)\right)^{\frac{q}{p}} \|F\|_s \\ &\le \|f\|_p^q \|F\|_s \qquad (5.1)\end{aligned}$$

using Hölder's inequality with conjugate exponents p/q and s.
For $z \neq 0,\ z \in \mathbb{D}$, let $\tilde{I}(z) = \{\zeta \in \mathbb{T} \mid \ z \in \Gamma(\zeta)\}$. In other words $\zeta \in \tilde{I}(z) \Leftrightarrow z \in \Gamma(\zeta)$. Then

$$m\left(\tilde{I}(z)\right) \approx 1 - |z| \qquad (5.2)$$

and

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q \, d\mu_r(z) &\approx \int_{\mathbb{D}} |f(z)|^q \left(\int_{\tilde{I}(z)} dm(\zeta)\right) \frac{d\mu_r(z)}{1-|z|^2} \\ &= \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)|^q \frac{d\mu_r(z)}{1-|z|^2}\, dm(\zeta) \\ &\le \int_{\mathbb{T}} Mf(\zeta)^q \int_{\Gamma(\zeta)} \frac{d\mu_r(z)}{1-|z|^2}\, dm(\zeta)\end{aligned}$$

where $Mf(\zeta) = \sup_{z\in\Gamma(\zeta)} |f(z)|$ is the Hardy-Littlewood maximal nontangential function. We apply Hölder's inequality to obtain

$$\int_{\mathbb{D}} |f(z)|^q \, d\mu_r(z) \le \gamma \|Mf\|_p^q \|\widetilde{G_r}\|_s \le \gamma \|M\|_p^q \|\widetilde{G_r}\|_s \|f\|_p^q, \qquad (5.3)$$

where γ is a positive constant that appears in (5.2). Combining (5.1) and (5.3) it follows that

$$\int_{\overline{\mathbb{D}}} |f(z)|^q \, d\mu_r(z) \le (\|F\|_s + \gamma \|M\|_p^q \|\widetilde{G_r}\|_s) \|f\|_p^q.$$

It remains to show that $\|\widetilde{G_r}\|_s \to 0$ when $r \to 1$. We will make use of Lebesgue's dominated convergence theorem. Clearly we have $0 \le \widetilde{G_r} \le G \in L^s(\mathbb{T})$, so we need to show that $\widetilde{G_r}(\zeta) \to 0$ as $r \to 1$ for $m-$almost every $\zeta \in \mathbb{T}$. Let $A = \{\zeta \in \mathbb{T} \mid \ G(\zeta) < \infty\}$. It is a set of full measure ($m(A) = 1$) since $G \in L^s(\mathbb{T})$. Write $\widetilde{G_r}(\zeta) = \int_{\Gamma(\zeta)} \tilde{f}_r(z)\, d\mu(z)$ with $\tilde{f}_r(z) = \mathbb{1}_{\overline{\mathbb{D}}\setminus r\mathbb{D}}(z)(1 - |z|^2)^{-1},\ z \in \Gamma(\zeta)$. For every $\zeta \in A$ one has

$$\begin{aligned}&\left|\tilde{f}_r(z)\right| \le \frac{1}{1-|z|^2} \in L^1\left(\Gamma(\zeta), \mu\right) \text{ since } \zeta \in A, \\ &\tilde{f}_r(z) \underset{r\to 1}{\longrightarrow} 0 \text{ for all } z \in \Gamma(\zeta) \subset \mathbb{D}.\end{aligned}$$

Lebesgue's dominated convergence theorem in $L^1(\Gamma(\zeta), \mu)$ ensures that $\widetilde{G_r}(\zeta) = \|\tilde{f}_r\|_{L^1(\Gamma(\zeta),\mu)}$ tends to zero as r tends to 1 for $m-$almost every $\zeta \in \mathbb{T}$, which ends the proof. □

Theorem 5.4. *Let u be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Assume that uC_φ is a bounded operator from H^p to H^q, with $\infty > p > q \geq 1$. Then*

$$\|uC_\varphi\|_e \leq 2\|C_\varphi\|_{p/q}^{1/q} \left(\int_{E_\varphi} |u(\zeta)|^{\frac{pq}{p-q}} \, \mathrm{d}m(\zeta) \right)^{\frac{p-q}{pq}},$$

where $\|C_\varphi\|_{p/q}$ denotes the norm of C_φ acting on $H^{p/q}$.

Proof. We follow the same lines as in the proof of the upper estimate in Proposition 2.4: we have the decomposition $I = K_N + R_N$ in $B(H^p)$, where K_N is the convolution operator by the Fejér kernel, and

$$\|uC_\varphi\|_e \leq \liminf_N \|uC_\varphi R_N\|.$$

We also have, for every $0 < r < 1$,

$$\begin{aligned}
\|(uC_\varphi R_N)f\|_q^q &= \int_{\overline{\mathbb{D}}\setminus r\mathbb{D}} |R_N f(w)|^q \, \mathrm{d}\mu_\varphi(w) + \int_{r\mathbb{D}} |R_N f(w)|^q \, \mathrm{d}\mu_\varphi(w) \\
&= I_1(N, r, f) + I_2(N, r, f).
\end{aligned}$$

As in the $p \leq q$ case, we show that

$$\lim_N \sup_{\|f\|_p \leq 1} I_2(N, r, f) = 0.$$

The measure μ_φ being a (p, q)-Carleson measure, we use Lemma 5.3 to have the following inequality

$$I_1(N, r, f) \leq (\|F\|_s + \gamma\|M\|_p^q \|\widetilde{G_r}\|_s)\|R_N f\|_p^q$$

for every $f \in H^p$. As a consequence

$$\|uC_\varphi\|_e \leq \liminf_N \left(\sup_{\|f\|_p \leq 1} I_1(N, r, f) \right)^{\frac{1}{q}} \leq 2(\|F\|_s + \gamma\|M\|_p^q \|\widetilde{G_r}\|_s)^{\frac{1}{q}}$$

using the fact that $\sup_N \|R_N\| \leq 2$. Now we make r tend to 1, keeping in mind that $\|\widetilde{G_r}\|_s \to 0$. We obtain

$$\|uC_\varphi\|_e \leq 2\|F\|_s^{1/q}.$$

It remains to see that we can choose F in such a way that

$$\|F\|_s \leq \|C_\varphi\|_{p/q} \left(\int_{E_\varphi} |u(\zeta)|^{\frac{pq}{p-q}} \; \mathrm{d}m(\zeta) \right)^{1/s}.$$

Indeed, if $f \in C(\mathbb{T}) \cap H^{p/q}$, we apply Hölder's inequality with conjugates exponents p/q and s to have

$$\begin{aligned}
\left| \int_{\mathbb{T}} f \; \mathrm{d}\mu_{\varphi,\mathbb{T}} \right| &= \left| \int_{E_\varphi} |u|^q f \circ \varphi \; \mathrm{d}m \right| \\
&\leq \int_{E_\varphi} |u|^q |f \circ \varphi| \; \mathrm{d}m \\
&\leq \|C_\varphi(f)\|_{p/q} \left(\int_{E_\varphi} |u|^{sq} \; \mathrm{d}m \right)^{1/s},
\end{aligned}$$

meaning that $\mu_{\varphi,\mathbb{T}} \in \left(H^{p/q}\right)^*$, which is isometrically isomorphic to $L^s(\mathbb{T})/H^s_0$, where H^s_0 is the subspace of H^s consisting of functions vanishing at zero. If we denote by $N(\mu_{\varphi,\mathbb{T}})$ the norm of $\mu_{\varphi,\mathbb{T}}$ viewed as an element of $\left(H^{p/q}\right)^*$, then one can choose $F \in L^s(\mathbb{T})$ satisfying

$$\|F\|_s = N(\mu_{\varphi,\mathbb{T}}) \leq \|C_\varphi\|_{p/q} \left(\int_{E_\varphi} |u|^{pq/(p-q)} \; \mathrm{d}m \right)^{1/s}$$

and $\mu_{\varphi,\mathbb{T}} = F \; \mathrm{d}m$ (see for instance [11], p. 194). Finally we have

$$\|uC_\varphi\|_e \leq 2\|C_\varphi\|_{p/q}^{1/q} \left(\int_{E_\varphi} |u(\zeta)|^{\frac{pq}{p-q}} \; \mathrm{d}m(\zeta) \right)^{\frac{p-q}{pq}}.$$

$\square$

Although we have not be able to give a corresponding lower bound of this form for the essential norm of uC_φ, we have the following result:

Proposition 5.5. *Let $1 \leq q < p < \infty$, and assume that $uC_\varphi \in B(H^p, H^q)$. Then*

$$\|uC_\varphi\|_e \geq \left(\int_{E_\varphi} |u(\zeta)|^q \; \mathrm{d}m(\zeta) \right)^{\frac{1}{q}}.$$

Proof. Take a compact operator K from H^p to H^q. Since it is completely continuous, and the sequence (z^n) converges weakly to zero in H^p, $(K(z^n))_n$ converges to zero in H^q. Hence

$$\|uC_\varphi + K\| \geq \|(uC_\varphi + K)z^n\|_q \geq \|uC_\varphi(z^n)\|_q - \|K(z^n)\|_q$$

for every $n \geq 0$. Taking the limit as n tends to infinity, we have

$$\|uC_\varphi\|_e \geq \left(\int_{E_\varphi} |u(\zeta)|^q \, \mathrm{d}m(\zeta)\right)^{\frac{1}{q}}.$$

□

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References

[1] O. Blasco and H. Jarchow. *A note on Carleson measures for Hardy spaces.* Acta Sci. Math. (Szeged) **71** (2005), 371–389.

[2] L. Carleson *Interpolations by bounded analytic functions and the corona problem.* Ann. of Math. (2) **76** (1962), 547–559.

[3] M. D. Contreras and A. G. Hernández-Díaz *Weighted Composition Operators between Different Hardy Spaces.* Integr. eq. oper. theory **46** (2003), 165–188.

[4] C. C. Cowen and B. D. MacCluer. *Composition operators on spaces of analytic functions.* CRC Press, Boca Raton, 1995.

[5] Ž. Čučković and R. Zhao. *Weighted composition operators between different weighted Bergman spaces and different Hardy spaces.* Illinois J. Math. **51** no. 2 (2007), 479–498.

[6] Ž. Čučković and R. Zhao. *Weighted composition operators on the Bergman spaces.* J. London Math. Soc. (2) **51** (2004), 499–511.

[7] P. Duren. *Theory of H^p spaces.* Academic Press, New York, 1970.

[8] F. Forelli. *The isometries of H^p.* Canad. J. Math. **16** (1964), 721–728.

[9] P. Gorkin and B. D. MacCluer. *Essential Norms of Composition Operators.* Integr. eq. oper. theory **48** (2004), 27–40.

[10] K. Hoffman. *Banach spaces of analytic functions.* Dover Publications, Inc., 1988

[11] P. Koosis. *Introduction to H_p spaces.* Cambridge University Press, 1980.

[12] P. Lefèvre. *Generalized Essential Norm of Weighted Composition Operators on some Uniform Algebras of Analytic Functions.* Integr. eq. oper. theory **63** (2009), 557–569.

[13] V. Matache. *Composition operators on Hardy Spaces of a half-plane.* Proc. Amer. Math. Soc. **127** (1999), 1483–1491.

[14] S. D. Sharma and R. K. Singh. *Composition operators on Hardy spaces of the upper half-plane.* Bull. Allahabad Math. Soc. **14** (1999), 129–145.

[15] K. Zhu. *Duality of Bloch Spaces and Norm Convergence of Taylor Series.* Michigan Math. J. **38** (1991), no. 1, 89–101.